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ON AVAILABILITY IN COMPOUND-POISSON DEMAND INVENTORY SYSTEMS

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ABSTRACT

Three possible performance measures in compound-Poisson demand inventory systems can be used to describe the availability of stock to a customer. Simple relationship between these three availabilities are derived under general assumptions about the inventory process.

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Consider a single-item inventory system in which the demand is compound-Poisson. There are three possible performance measures which might be used to describe the availability of stock to a customer:

1. α_t , the *temporal availability*, is the (long-run) fraction of time that the inventory is positive; and
2. α_i , the *item availability*, is the fraction of total demand which can be filled immediately from stock (or can ever be filled, if back ordering is not allowed).
3. α_b , the *batch availability*, is the fraction of batch orders which can be filled completely and immediately from stock (or can ever be filled).

In queueing terminology, the first two might also be called the *virtual* and *actual availabilities*, respectively. Professor E. A. Silver suggested to the author that there might be a simple relationship between these first two measures, and independently derived Equation (10) under special assumptions[3]. The purpose of this note is to develop simple relationships between all three availabilities under quite general assumptions about the nature of the inventory system.

To fix notation, let the Poisson *batch demand* parameter be λ_o batches/day, and let ν be the random *batch order size*, with:

(1) $$p_j = P\{\nu = j\} \qquad (j = 1,2,3) \ ;$$

we assume the first two moments, $E\{\nu\}$ and $V\{\nu\}$, are finite. Then total *item order demand* flows in at an average rate

(2) $$\lambda = \lambda_o E\{\nu\} \qquad \text{items/day.}$$

Consider a typical realization of the stock level of the inventory system shown in Figure 1. The nth *cycle* begins just after the nth replenishment when, the *state of the system* is χ_n ; the current cylce lasts an *interval* τ_n , until the next replenishment to state χ_{n+1} . During this interval, the compound Poisson demands deplete the stock level, until (possibly) some order causes the stock level to "under shoot" the zero level by a *deficit*, δ_n . Further demands may cause the inventory position to further degrade to a negative level, $-\epsilon_n$, before the arrival of a replenishment lot, ξ_n , concludes this cycle; the total *shortage interval* is σ_n .

[In the event that physical backlog is not allowed, the negative stock levels in Figure 1 refer to orders which are never filled. ξ_n is interpreted as the actual physical reorder quantity *plus* ϵ_n , the total lost sales for that cycle.]

The only assumption we shall make about the inventory operating policy, including ordering mechanisms, safety stock, filling priority, multiple and/or emergency replenishments, etc., etc. is that *the stochastic process, state of the system, is a Markov-renewal process*, i.e., the nontime-varying conditional probabilities

$$(3)\qquad P\{\chi_{n+1} = y \; ; \; \xi_n = q \; ; \; \epsilon_n = e \; ; \; \delta_n = d \; ; \; \sigma_n = s \; ; \; \tau_n = t \mid \chi_n = x\}$$

are sufficient to determine the evolution of the system, once the initial state is known. The generality of this assumption can be appreciated when we emphasize that χ_n , the state of the system at the beginning of the nth cycle, may include not only the physical inventory level, but various *supplementary variables*, such as the current distributions of times-of-arrival and sizes of outstanding replenishments.

Naturally, we require that a stationary distribution of the process exist. A necessary condition for this to occur, back orders or not, is:

$$(4)\qquad E\{\xi\} = \lambda E\{\tau\} .$$

(We drop subscripts for an arbitrary cycle, and take all expectations using the transition probabilities (3) and the appropriate stationary probabilities.) For convenience in the sequel, we assume $E\{\tau\}$ is finite.

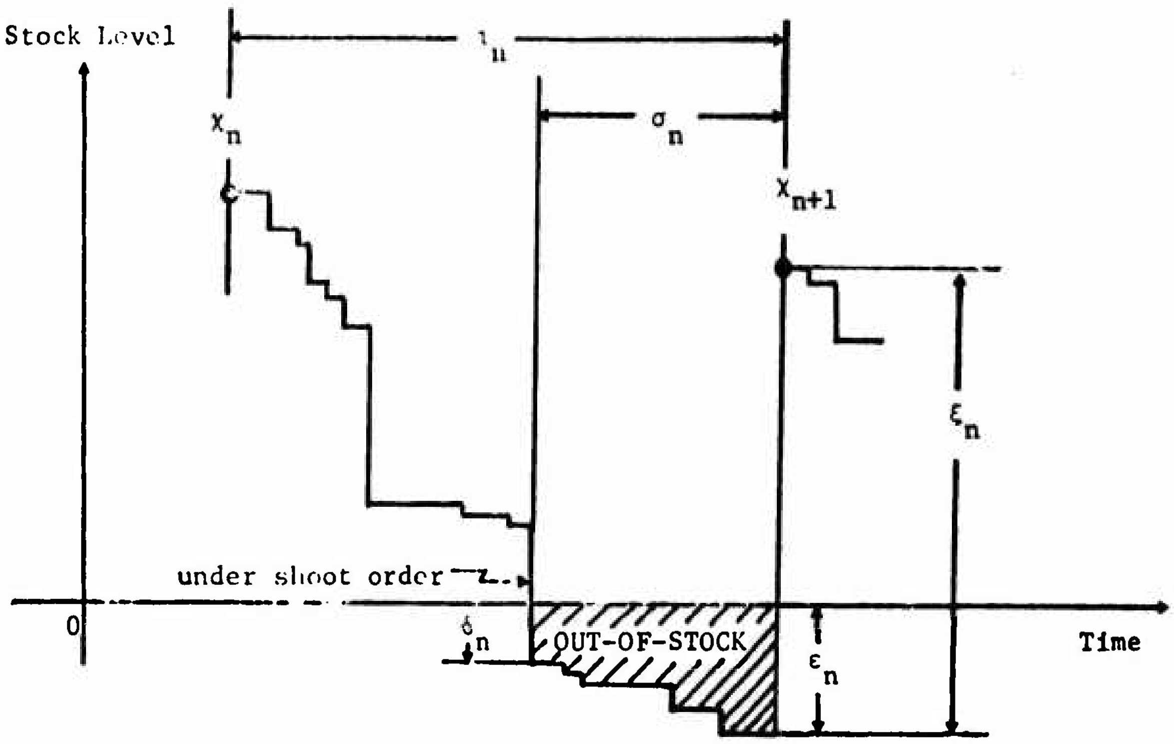

FIGURE 1: TYPICAL REALIZATION OF STOCK LEVEL FOR GENERAL INVENTORY SYSTEM WITH COMPOUND-POISSON DEMAND

By considering σ as a *cumulative renewal function* over the Markov-renewal process [2], we may use a generalized form of the Renewal Theorem to find, for example, the limiting cumulative average rate at which shortage (out-of-stock) intervals are accumulated.

$$(5) \qquad \lim_{t\to\infty} \frac{E\{\text{Cumulative total of shortage intervals in horizon } (0,t]\}}{t} = \frac{E\{\sigma\}}{E\{\tau\}} .$$

Here $E\{\sigma\}$ *includes* those cycles in which $\sigma = 0$. Since (5) is just the fraction of time the inventory is nonpositive, the temporal availability is:

$$(6) \qquad \alpha_t = 1 - \frac{E\{\sigma\}}{E\{\tau\}} .$$

Finding these expectations, however, may be a formidable task in an inventory system of any reasonable complexity.

Similar remarks apply to, ε_n , the items in any cycle which are not filled immediately (or are never filled).

$$(7) \qquad \lim_{t\to\infty} \frac{E\{\text{Cumulative number of items not filled immediately in horizon } (0,t]\}}{t} = \frac{E\{\varepsilon\}}{E\{\tau\}} .$$

Since items are demanded at rate λ , it follows that item availability is:

$$(8) \qquad \alpha_i = 1 - \frac{E\{\varepsilon\}}{\lambda E\{\tau\}} .$$

Because the demand is compound Poisson, we note that

$$(9) \qquad E\{\varepsilon\} = E\{\delta\} + \lambda E\{\sigma\} .$$

As before, expectations include those cycles where δ , ε , and/or σ are zero.

Using (4), (6) and (8), the item availability is:

(10) $$\alpha_i = \alpha_t - \frac{E\{\delta\}}{\lambda E\{\tau\}} = \alpha_t - \frac{E\{\delta\}}{E\{\xi\}} \; .$$

Assuming one of the availabilities is known, the problem then reduces to finding the fraction: (average deficit per cycle/average replenishment quantity).

For batch availability, we use similar arguments to find:

(11) $$\lim_{t\to\infty} \frac{E\{\text{Cumulative number of batches not filled immediately in horizon } (0,t]\}}{t} = \frac{P\{\delta \geq 1\} + \lambda_o E\{\sigma\}}{E\{\tau\}} \; .$$

But since batches are demanded at average rate λ_o , we find:

(12) $$\alpha_b = \alpha_t - \frac{P\{\delta \geq 1\}}{(E\{\xi\}/E\{\nu\})} \; .$$

We recognize the correction term as the fraction (probability a batch is "broken" per cycle/average number of batches per replenishment).

We note trivially that if we have unit order size, $\delta = 0$ always, and (10) and (12) simplify to:

(13) $$\alpha_t = \alpha_i = \alpha_b \; ,$$

which checks with known results for simple Poisson-demand inventory models.

More generally, the problem of determining the distribution of δ , given the inventory level at the beginning of the cycle, is similar to that of determining an *origin-dependent excess distribution* (see, for example [1]) in the renewal theory, where the order-size distribution, $\{p_j\}$, corresponds to a discrete inter-event distribution. This excess distribution must then be weighted by the stationary distribution of the maximum stock-level (the "origin").

In most practical applications, however, ν assumes a large number of different values, and either $E\{\xi - \varepsilon\} >> E\{\nu\}$, or the distribution of highest inventory level is fairly well "smeared out" over a range of states large compared to $E\{\nu\}$. Under this condition, it is reasonable to assume that the distribution of δ, given that a stock-out occurs, is just the *equilibrium excess distribution*[1] corresponding to $\{p_j\}$, i.e.,

$$(14) \qquad P\{\delta = j \mid \text{stock-out}\} = \frac{1}{E\{\nu\}} \sum_{i=j+1}^{\infty} p_i \qquad (j = 0,1,2, \ldots) .$$

Let π_o be the probability that a stock-out occurs in an arbitrary cycle. Then

$$(15) \qquad E\{\delta\} = \pi_o\left[E\{\nu\}\left(\frac{k^2 + 1}{2}\right) - \tfrac{1}{2}\right] ,$$

and

$$(16) \qquad P\{\delta \geq 1\} = \pi_o\left[\frac{E\{\nu\} - 1}{E\{\nu\}}\right] ,$$

where

$$(17) \qquad k^2 = V\{\nu\}/[E\{\nu\}]^2$$

is the *coefficient of variation of order size.* (15) and (16) may then be used to simplify (10) and (12) so that we have, finally:

$$(18) \qquad \alpha_i = \alpha_t - \frac{\pi_o}{E\{\xi\}}\left[\frac{E\{\nu\}(k^2 + 1) - 1}{2}\right]$$

$$(19) \qquad \alpha_b = \alpha_t - \frac{\pi_o}{E\{\xi\}}\left[E\{\nu\} - 1\right] .$$

If

$$(20) \qquad k^2 > 1 - (E\{\nu\})^{-1}$$

then $\alpha_i < \alpha_b$. Of course $\alpha_i \leq \alpha_t$ and $\alpha_b \leq \alpha_t$ always.

Since α_t will usually be monotone in the decision variables of the system [say, average replenishment lot size], as will $-\pi_o/E\{\xi\}$, it follows that α_i and α_b will also be monotone. Thus, for comparison of operating policies, there seems to be little difference as to whether temporal, item or batch availability is used to measure system performance. However, in practical situations, the correction terms in (18) and (19) may be significant [3].

Finally, we note that all of these results apply to performance measures calculated when passing below any arbitrary stock level.

I would like to thank E. A. Silver for his comments on the first draft.

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